THE SOLUTION OF THE NONLINEAR HEAT-CONDUCTION PROBLEM FOR BOUNDARY CONDITIONS OF THE FOURTH KIND

A. A. Aleksashenko and V. A. Aleksashenko

Inzhenerno-Fizicheskii Zhurnal, Vol. 13, No. 5, pp. 754–758, 1967

UDC 536.2.01

Certain exact solutions have been derived for the nonlinear heat-conduction problem for boundary conditions of the fourth kind.

On the basis of the method proposed by the authors in [1], we have derived exact solutions for the heat-conduction problem under boundary conditions of the fourth kind. The thermophysical parameters are functions of temperature.

§1. Let us examine the system of equations

$$c_1(\theta_1)\gamma_1(\theta_1)\frac{\partial\theta_1}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_1(\theta_1)\frac{\partial\theta_1}{\partial x}\right) \quad (t>0,\quad x>0), \tag{1.1}$$

$$c_2(\theta_2)\gamma_2(\theta_2)\frac{\partial\theta_2}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_2(\theta_2)\frac{\partial\theta_2}{\partial x}\right) \quad (t>0,\quad x<0) \tag{1.2}$$

for the following boundary conditions:

$$\theta_1(x,\ 0)=\theta_{10}\quad (x>0),\qquad \theta_2(x,\ 0)=\theta_{20}\quad (x<0); \tag{1.3}$$

$$\theta_1(0,\ t)=\theta_2(0,\ t),\quad \lambda_1\frac{\partial\theta_1(0,\ t)}{\partial x}=\lambda_2\frac{\partial\theta_2(0,\ t)}{\partial x}, \tag{1.4}$$

where θ_{10} and θ_{20} are constants.

We assume that

$$\xi=\frac{|x|}{2\sqrt{t}}, \tag{1.5}$$

so that with consideration of (1.5) Eqs. (1.1) and (1.2) assume the form

$$\frac{d}{d\xi}\left[\lambda_1(\theta_1)\frac{d\theta_1}{d\xi}\right]=-2\xi\frac{d\theta_1}{d\xi}c_1(\theta_1)\gamma(\theta_1), \tag{1.6}$$

$$\frac{d}{d\xi}\left[\lambda_2(\theta_2)\frac{d\theta_2}{d\xi}\right]=-2\xi\frac{d\theta_2}{d\xi}c_2(\theta_2)\gamma(\theta_2), \tag{1.7}$$

which we will solve for the boundary conditions

$$[\theta_1]_{\xi=\infty}=\theta_{10},\quad [\theta_2]_{\xi=\infty}=\theta_{20}; \tag{1.8}$$

$$[\theta_1]_{\xi=0}=[\theta_2]_{\xi=0},\quad \lambda_1\left[\frac{d\theta_1}{d\xi}\right]_{\xi=0}=-\left[\frac{d\theta_2}{d\xi}\right]_{\xi=0}\lambda_2. \tag{1.9}$$

§2. Let us indicate the method of solving equations of the form of (1.6) and (1.7) (we drop the subscripts).

Let us introduce the substitution

$$u=\int_{\theta_0}^{\theta}\lambda(\theta)\,d\theta+\alpha, \tag{2.1}$$

where α is a constant equal to the value of the antiderivative at the lower limit.

With (2.1) we modify (1.6) to the form

$$\frac{d^2u}{d\xi^2}=-2\xi\frac{du}{d\xi}f(u), \tag{2.2}$$

where

$$f(u)=c(\theta)\gamma(\theta)/\lambda(\theta)=\bar{c}(u)\,\bar{\gamma}(u)/\bar{\lambda}(u).$$

By means of the substitution

$$du/d\xi=\varphi(u) \tag{2.3}$$

we reduce (2.2) to the form

$$\frac{d\varphi}{du}F(u)=-2\xi, \tag{2.4}$$

where

$$F(u)=[f(u)]^{-1}. \tag{2.5}$$

Differentiating with respect to ξ in (2.4) and assuming that

$$y=\varphi\sqrt{F(u)}, \tag{2.6}$$

we obtain

$$y''+I(u)y=-2/y, \tag{2.7}$$

where

$$I(u)=\frac{1}{4}\{2[\ln f(u)]''-([\ln f(u)]')^2\}. \tag{2.8}$$

§3. Let us investigate Eq. (2.7).

Let us assume

$$I(u)=\beta, \tag{3.1}$$

where β is a constant.

Let

$$\beta\neq 0. \tag{3.2}$$

Equation (2.7) then assumes the form

$$y''+\beta y=-2/y. \tag{3.3}$$

Its solution is

$$u=\int_0^y\frac{dy'}{\sqrt{D-4\ln y'-\beta y'^2}}+A, \tag{3.4}$$

where D and A are constants.

Having solved (3.1) with consideration of (3.2), we find that the function $f(u)$ can assume the following form:

$$f(u)=R\exp\left[\pm\sqrt{-\beta}\,u\right], \tag{3.5}$$

$$f(u)=\frac{M}{\cos^2\left[\sqrt{\beta}\,(u+K)\right]}\quad(\beta>0), \tag{3.6}$$

$$f(u)=\frac{N}{\operatorname{ch}^2\left[\sqrt{-\beta}\,(u+K)\right]}$$

$$(\beta<0,\quad |\operatorname{th}[\sqrt{-\beta}\,(u+K)]|<1), \tag{3.7}$$

$$f(u)=\frac{N}{\operatorname{sh}^2\left[\sqrt{-\beta}\,(u+K)\right]}$$

$$(\beta<0,\quad |\operatorname{th}[\sqrt{-\beta}\,(u+K)]|>1), \tag{3.8}$$

where R, M, N, and K are constants.

Let

$$\beta=0. \tag{3.9}$$

In this case solution (3.3) is derived from (3.4) with consideration of (3.9).

Solution (3.1) with consideration of (3.9) yields

$$f(u)=B, \tag{3.10}$$

$$f(u)=a/(u+b)^2, \tag{3.11}$$

where B, a, and b are constants.

§4. Consequently, for cases (3.5)–(3.8) and (3.10) and (3.11) we have derived a solution in quadratures in the form of (3.4).

From the systems of equations

$$c(\theta)\,\gamma(\theta)/\lambda(\theta)=f(u),$$

$$u=\int_{\theta_0}^{\theta}\lambda(\theta)\,d\theta+\alpha, \tag{4.1}$$

eliminating u, we find the form for the variable thermophysical parameters for cases (3.5)–(3.8) and (3.10) and (3.11), arbitrarily specifying the form of any two parameters.

The formula

$$c(\theta)\,\gamma(\theta)=\lambda(\theta)\,f\left[\int_{\theta_0}^{\theta}\lambda(\theta)\,d\theta+\alpha\right] \tag{4.2}$$

is convenient to find $c(\theta)\gamma(\theta)$ from the given $\lambda(\theta)$; however, it is not convenient to find $\lambda(\theta)$ from $c(\theta)$ and $\gamma(\theta)$. In the latter case differentiation in (4.2) should be employed to derive the equation for the determination of $\lambda(\theta)$.

§5. Solutions (1.6) and (1.7), respectively, have the forms

$$u_1=\int_0^{y_1}\frac{dy}{\sqrt{D_1-4\ln y-\beta_1 y^2}}+A_1, \tag{5.1}$$

$$u_2=\int_0^{y_2}\frac{dy}{\sqrt{D_2-4\ln y-\beta_2 y^2}}+A_2. \tag{5.2}$$

Boundary conditions (1.8) and (1.9), with consideration of (2.1), assume the forms

$$[u_1]_{\xi=\infty}=\alpha_1, \tag{5.3}$$

$$[u_2]_{\xi=\infty}=\alpha_2, \tag{5.4}$$

$$[u_1]_{\xi=0}=\int_{\theta_{10}}^{[\theta_1]_{\xi=0}}\lambda_1(\theta)\,d\theta+\alpha_1=\Phi_1([\theta_1]_{\xi=0}), \tag{5.5}$$

$$[u_2]_{\xi=0}=\int_{\theta_{20}}^{[\theta_2]_{\xi=0}}\lambda_2(\theta)\,d\theta+\alpha_2=\Phi_2([\theta_2]_{\xi=0}), \tag{5.6}$$

where $\Phi_1(\theta_1)$ and $\Phi_2(\theta_2)$ are the antiderivatives of the functions in (5.5) and (5.6), respectively. We denote

$$[u_1]_{\xi=0}=m,\quad [u_2]_{\xi=0}=n, \tag{5.7}$$

where m and n are constants.

From (5.5) and (5.6), with consideration of (5.7), we find

$$[\theta_1]_{\xi=0}=\Psi_1[m],$$

$$[\theta_2]_{\xi=0}=\Psi_2[n],$$

and, substituting into (1.9), we obtain

$$\Psi_1[m]=\Psi_2[n]. \tag{5.8}$$

The second condition in (1.4), with consideration of (2.1), (2.4) and (2.6), assumes the form

$$\left(\frac{y_1}{\sqrt{F_1(u_1)}}\right)_{u_1=m}=-\left(\frac{y_2}{\sqrt{F_2(u_2)}}\right)_{u_2=n}. \tag{5.9}$$

The constants in (5.1) and (5.2) are found from (5.3), (5.4) and (5.8), (5.9). Since

$$\left[\frac{du_1}{d\xi}\right]_{\xi=\infty}=[\varphi_1(u_1)]_{u_1=\alpha_1}=0, \tag{5.10}$$

$$\left[\frac{du_2}{d\xi}\right]_{\xi=\infty}=[\varphi_2(u_2)]_{u_2=\alpha_2}=0, \tag{5.11}$$

and it follows from (2.4) that

$$\left[\frac{d\varphi_1}{du_1}\right]_{u_1=m}=\left[\frac{d\varphi_2}{du_2}\right]_{u_2=n}=0 \tag{5.12}$$

(we assume that $[F_1(u_1)]_{u_1=m}\neq 0$ and $[F_2(u_2)]_{u_2=n}\neq 0$), so that boundary conditions (5.1) and (5.2), with consideration of (2.6) and (5.10)–(5.11), are written as follows:

$$[y_1]_{u_1=\alpha_1}=[\varphi_1\sqrt{F_1(u_1)}]_{u_1=\alpha_1}=0, \tag{5.13}$$

$$[y_2]_{u_2=\alpha_2}=[\varphi_2\sqrt{F_2(u_2)}]_{u_2=\alpha_2}=0, \tag{5.14}$$

and since

$$\frac{d\varphi_1}{du_1}=\frac{dy_1}{du_1}\,\frac{1}{\sqrt{F_1(u_1)}}-\frac{F_1'(u_1)\,y_1}{2F_1(u_1)\sqrt{F_1(u_1)}},$$

with consideration of (5.12) we obtain

$$\left[\frac{d\varphi_1}{du_1}\right]_{u_1=m}=\left[\frac{dy_1}{du_1}-\frac{F_1'(u_1)\,y_1}{2F_1(u_1)}\right]_{u_1=m}=0 \tag{5.15}$$

and analogously

$$\left[\frac{d\varphi_2}{du_2}\right]_{u_1=n} = \left[\frac{dy_2}{du_2} - \frac{F_2'(u_2)\,y_2}{2F_2(u_2)}\right]_{u_2=n} = 0. \qquad (5.16)$$

Substituting (5.1) and (5.2) into (5.13) and (5.14), respectively, we find that

$$A_1 = \alpha_1, \quad A_2 = \alpha_2. \qquad (5.17)$$

Substituting (5.1) and (5.2) into (5.15) and (5.16), respectively, we derive a system of equations from which we define the constants D_1 and D_2:

$$\left(\sqrt{D_1 - 4\ln y_1 - \beta_1 y_1^2} - \frac{F_1'(u_1)\,y_1}{2F_1(u_1)}\right)_{u_1=m} = 0,$$

$$\left(\sqrt{D_2 - 4\ln y_2 - \beta_2 y_2^2} - \frac{F_2'(u_2)\,y_2}{2F_2(u_2)}\right)_{u_2=n} = 0,$$

$$\Psi_1(m) = \Psi_2(n),$$

$$\left(\frac{y_1}{\sqrt{F_1(u_1)}}\right)_{u_1=m} = -\left(\frac{y_2}{\sqrt{F_2(u_2)}}\right)_{u_2=n},$$

$$u_1 = \int_0^{y_1} \frac{dy}{\sqrt{D_1 - 4\ln y - \beta_1 y^2}} + \alpha_1,$$

$$u_2 = \int_0^{y_2} \frac{dy}{\sqrt{D_2 - 4\ln y - \beta_2 y^2}} + \alpha_2,$$

$$[u_1]_{\xi=0} = m, \quad [u_2]_{\xi=0} = n. \qquad (5.18)$$

After we have found the constants from system (5.18), from (5.1) and (5.2)—considering (2.3) and (2.6)—we find u_1 and u_2 as functions of ξ. Finally, we obtain an answer in the form

$$\theta_1(x, t) = \Psi_1[u_1], \quad \theta_2(x, t) = \Psi_2[u_2]. \qquad (5.19)$$

REFERENCES

1. A. A. Aleksashenko, V. A. Aleksashenko, and N. V. Seleznev, collection: Structural Thermophysics [in Russian], izd. Energiya, 1966.

4 October 1966

Zhukovskii Central Aerohydrodynamics Institute, Moscow